

# Average Size of a Self-conjugate $(s, t)$ -Core Partition

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## Abstract

Armstrong, Hanusa and Jones conjectured that if  $s, t$  are coprime integers, then the average size of an  $(s, t)$ -core partition and the average size of a self-conjugate  $(s, t)$ -core partition are both equal to  $\frac{(s+t+1)(s-1)(t-1)}{24}$ . Stanley and Zanello showed that the average size of an  $(s, s+1)$ -core partition equals  $\binom{s+1}{3}/2$ . Based on a bijection of Ford, Mai and Sze between self-conjugate  $(s, t)$ -core partitions and lattice paths in  $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$  rectangle, we obtain the average size of a self-conjugate  $(s, t)$ -core partition as conjectured by Armstrong, Hanusa and Jones.

**Keywords:**  $(s, t)$ -core partition, self-conjugate partition, lattice path

**AMS Classification:** 05A17, 05A15

## 1 Introduction

In this paper, employing a bijection of Ford, Mai and Sze between self-conjugate  $(s, t)$ -core partitions and lattice paths, we prove a conjecture of Armstrong, Hanusa and Jones on the average size of a self-conjugate  $(s, t)$ -core partition.

A partition is called a  $t$ -core partition, or simply a  $t$ -core, if its Ferrers diagram contains no cells with hook length  $t$ . A partition is called an  $(s, t)$ -core partition, or simply an  $(s, t)$ -core, if it is simultaneously an  $s$ -core and a  $t$ -core. When  $\gcd(s, t) = r > 1$ , each  $r$ -core is an  $(s, t)$ -core, which means that there are infinitely many  $(s, t)$ -cores. When  $s$  and  $t$  are coprime, Anderson [1] showed that the number of  $(s, t)$ -core partitions equals

$$\frac{1}{s+t} \binom{s+t}{s}.$$

Under the same condition, Ford, Mai and Sze [4] characterized the set of hook lengths of diagonal cells in self-conjugate  $(s, t)$ -core partitions, and they showed that the number of self-conjugate  $(s, t)$ -core partitions is

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}. \quad (1.1)$$

A partition is of size  $n$  if it is a partition of  $n$ . Aukerman, Kane and Sze [3] conjectured that the largest size of an  $(s, t)$ -core partition for  $s$  and  $t$  are coprime. Olsson and Stanton [5] proved this conjecture and gave the following stronger theorem.

**Theorem 1.1** *If  $s$  and  $t$  are coprime, then there is a unique largest  $(s, t)$ -core partition (which happens to be self-conjugate) of size*

$$\frac{(s^2 - 1)(t^2 - 1)}{24}. \quad (1.2)$$

A short proof for the conjecture of Aukerman, Kane and Sze was given by Tripathi [7]. Vandehey [8] gave the following characterization of the largest  $(s, t)$ -core partition.

**Theorem 1.2** *There exists a largest  $(s, t)$ -core partition  $\lambda$  under the partial order of containment. That is, for each  $(s, t)$ -core  $\mu$ ,  $\lambda_i \geq \mu_i$  for  $1 \leq i \leq l(\mu)$ .*

It is clear that the largest  $(s, t)$ -core in the above theorem is unique. It is the  $(s, t)$ -core of the largest size, and it is also a  $(s, t)$ -core of the longest length.

Recently, Armstrong, Hanusa and Jones [2] proposed the following conjecture concerning the average size of an  $(s, t)$ -core and the average size of a self-conjugate  $(s, t)$ -core.

**Conjecture 1.3** *Assume that  $s$  and  $t$  are coprime. Then the average size of an  $(s, t)$ -core and the average size of a self-conjugate  $(s, t)$ -core are both equal to*

$$\frac{(s + t + 1)(s - 1)(t - 1)}{24}.$$

Stanley and Zanello [6] showed that the conjecture for the average size of an  $(s, t)$ -core holds for  $(s, s + 1)$ -cores. More precisely, they obtained that the average size of an  $(s, s + 1)$ -core equals  $\binom{s+1}{3}/2$ . In this paper, we prove the Conjecture 1.3 pertaining to the average size of a self-conjugate  $(s, t)$ -core.

## 2 Proof of the conjecture for self-conjugate $(s, t)$ -cores

In this section, we prove the conjecture of Armstrong, Hanusa and Jones for self-conjugate  $(s, t)$ -cores. Let us begin with a quick review of the work on the structure of self-conjugate  $(s, t)$ -cores. Define

$$MD(\lambda) := \{h | h \text{ is the hook length of a cell on the main diagonal of } \lambda\}.$$

It is easily seen that a self-conjugate partition is uniquely determined by its main diagonal hooks. Ford, Mai and Sze [4] gave the following characterization of the main diagonal hook length set of a self-conjugate  $t$ -core  $\lambda$ .

**Theorem 2.1** *A self-conjugate partition  $\lambda$  is a  $t$ -core if and only if the following conditions hold:*

- (1) *if  $h \in MD(\lambda)$  and  $h > 2t$ , then  $h - 2t$  is also in  $MD(\lambda)$ ;*
- (2) *if  $h, s \in MD(\lambda)$ , then  $h + s \not\equiv 0 \pmod{2t}$ .*

69	53	37	21	5
47	31	15	-1	-17
25	9	-7	-23	-39
3	-13	-29	-45	-61

Figure 2.1: A lattice path in the array  $A(8, 11)$

To characterize the main diagonal hook lengths of a self-conjugate  $(s, t)$ -core, Ford, Mai and Sze [4] introduced an integer array  $A = (A_{i,j})_{1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor}$ , where

$$A_{i,j} = st - (2j - 1)s - (2i - 1)t, \quad (2.1)$$

for  $1 \leq i \leq \lfloor s/2 \rfloor$  and  $1 \leq j \leq \lfloor t/2 \rfloor$ . Let  $\mathcal{P}(A)$  be the set of lattice paths in  $A$  from the lower-left corner to the upper-right corner. For example, Figure 2.1 gives an array  $A$  for  $s = 8$  and  $t = 11$ , and the solid lines represent a lattice path in  $\mathcal{P}(A)$ . For a lattice path  $P$  in  $\mathcal{P}(A)$ , let  $M_A(P)$  denote the set of positive entries  $A_{i,j}$  below  $P$  and the absolute values of negative entries above  $P$ .

The following theorem is due to Ford, Mai and Sze [4].

**Theorem 2.2** *Assume that  $s$  and  $t$  are coprime. Let  $A$  be the array as given in (2.1). Then there is a bijection  $\Phi$  between the set  $\mathcal{P}(A)$  of lattice paths and the set of self-conjugate  $(s, t)$ -core partitions such that for  $P \in \mathcal{P}(A)$ , the set of main diagonal hook lengths of  $\Phi(P)$  is given by  $M_A(P)$ .*

For example, for the lattice path  $P$  in Figure 2.1, 5 is the only positive entry below  $P$ , while  $-7$  and  $-13$  are the negative entries above  $P$ . Thus  $M_A(P) = \{5, 7, 13\}$ . This gives  $\Phi(P) = (7, 5, 5, 3, 3, 1, 1)$ , which is an  $(8, 11)$ -core partition.

To compute the average size of self-conjugate  $(s, t)$ -cores, we show that the size of a partition  $\lambda$  can be expressed in terms of the entries in the array  $A$  above the lattice path  $P$  corresponding to  $\lambda$  under the bijection  $\Phi$ .

**Lemma 2.3** *For any lattice path  $P$  in  $\mathcal{P}(A)$ , we have*

$$|\Phi(P)| = \frac{(s^2 - 1)(t^2 - 1)}{24} - \sum_{(i,j) \text{ is above } P} A_{i,j}.$$

*Proof.* Clearly, the size of a self-conjugate partition equals the sum of the main diagonal hook lengths. By Theorem 2.2, we find that

$$\begin{aligned}
|\Phi(P)| &= \sum_{h \in MD(\Phi(P))} h \\
&= \sum_{(i,j) \text{ below } P, A_{i,j} > 0} A_{i,j} - \sum_{(i,j) \text{ is above } P, A_{i,j} < 0} A_{i,j} \\
&= \sum_{(i,j): A_{i,j} > 0} A_{i,j} - \sum_{(i,j) \text{ is above } P} A_{i,j}.
\end{aligned}$$

To show that

$$\sum_{(i,j): A_{i,j} > 0} A_{i,j} = \frac{(s^2 - 1)(t^2 - 1)}{24}, \quad (2.2)$$

let  $Q$  be the lattice path along the left and upper borders of  $A$ . Clearly,  $M_A(Q)$  consists of positive entries of  $A$ . Let  $\lambda = \Phi(Q)$ . By Theorem 2.2, the set of main diagonal hook length of  $\lambda$  equals  $M_A(Q)$ . Hence we obtain

$$|\lambda| = \sum_{(i,j): A_{i,j} > 0} A_{i,j}. \quad (2.3)$$

It remains to show that

$$|\lambda| = \frac{(s^2 - 1)(t^2 - 1)}{24}. \quad (2.4)$$

We claim that  $\lambda$  is the largest  $(s, t)$ -core. Thus (2.4) follows from the expression (1.2). To prove this claim, we recall that Theorem 1.1 guarantees that there is an  $(s, t)$ -core with largest size, say  $\mu$ , that happens to be self-conjugate. We aim to show that  $\mu = \lambda$ . Let  $l(\lambda)$  and  $l(\mu)$  denote the lengths of  $\lambda$  and  $\mu$  respectively. By Theorem 2.2, there is a lattice path  $R \in \mathcal{P}(A)$  such that  $\mu = \Phi(R)$ . By Theorem 1.2, we find that

$$l(\mu) \geq l(\lambda) \quad (2.5)$$

and

$$\mu_i \geq \lambda_i \quad (2.6)$$

for all  $i$ . Combining (2.5) and (2.6), we obtain that

$$\mu_1 + l(\mu) - 1 \geq \lambda_1 + l(\lambda) - 1. \quad (2.7)$$

The largest main diagonal hook length of  $\lambda$  is  $\lambda_1 + l(\lambda) - 1$ , that is,

$$\lambda_1 + l(\lambda) - 1 = \max MD(\lambda). \quad (2.8)$$

Since  $\lambda = \Phi(Q)$ , by Theorem 2.2, we have

$$MD(\lambda) = M_A(Q) = \{A_{i,j} | A_{i,j} > 0, 1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor\}. \quad (2.9)$$

Note that  $A_{1,1}$  is the largest in all positive entries in  $A$ . Thus, we deduce from (2.8) and (2.9) that

$$\lambda_1 + l(\lambda) - 1 = A_{1,1}. \quad (2.10)$$

Since  $\mu_1 + l(\mu) - 1$  is the hook length of the cell in the upper-left corner of  $\mu$ , by Theorem 2.2,  $\mu_1 + l(\mu) - 1$  belongs to  $M_A(R)$ . By the definition of  $M_A(R)$ , there exists an entry  $A_{i,j}$  of  $M_A(R)$  such that

$$\mu_1 + l(\mu) - 1 = |A_{i,j}|. \quad (2.11)$$

We claim that

$$A_{1,1} \geq |A_{i,j}|, \quad (2.12)$$

for any entry  $A_{i,j}$ . Note that  $A_{1,1}$  is the largest entry in  $A$ . On the other hand,  $A_{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor}$  is negative and is the smallest entry in  $A$ . It can be easily seen that  $|A_{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor}| < A_{1,1}$ , since

$$A_{1,1} + A_{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor} = st - s - t + st + s + t - 2t\lfloor s/2 \rfloor - 2s\lfloor t/2 \rfloor > 0.$$

This proves the claim.

Combining (2.10), (2.11) and (2.12), we obtain that

$$\lambda_1 + l(\lambda) - 1 \geq \mu_1 + l(\mu) - 1. \quad (2.13)$$

From (2.7) and (2.13), we deduce that

$$\lambda_1 + l(\lambda) - 1 = \mu_1 + l(\mu) - 1. \quad (2.14)$$

By (2.10) and (2.14), we see that  $A_{1,1} = \mu_1 + l(\mu) - 1$ , and hence it is a main diagonal hook length of  $\mu$ . Thus  $A_{1,1}$  lies in  $MD(\mu)$ . By Theorem 2.2,  $A_{1,1}$  belongs to  $M_A(R)$ . Since  $A_{1,1} > 0$ , it is an entry of  $A$  that is below the lattice path  $R$ . This implies that  $R$  is the unique lattice path of  $A$  along the left and upper borders. It follows that  $Q = R$  and  $\lambda = \mu$ . So we conclude that  $\lambda$  is the largest  $(s, t)$ -core. This completes the proof. ■

To prove the main result, we need some identities on the number of lattice paths in a rectangular region. Let  $m$  and  $n$  be positive integers, and  $B_{mn}$  be an  $m \times n$  diagram, that is, a diagram of  $m$  rows with each containing  $n$  cells. The positions of the cells of the first row are  $(1, 1), (1, 2), \dots, (1, n)$ , and so on. The set of lattice paths from the lower-left corner to the upper-right corner of  $B_{mn}$  is denoted by  $\mathcal{P}(B_{mn})$ . Let  $f(i, j)$  be the number of lattice paths in  $\mathcal{P}(B_{mn})$  that lie below the cell  $(i, j)$ , possibly touching the right or lower border of the cell  $(i, j)$ .

**Lemma 2.4** *For positive integers  $m, n$ , we have*

$$\sum_{1 \leq i \leq m, 1 \leq j \leq n} f(i, j) = \binom{m+n}{m} \frac{mn}{2}. \quad (2.15)$$

*Proof.* Given  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , the number of lattice paths in  $\mathcal{P}(B_{mn})$  below the cell  $(i, j)$  equals the number of lattice paths above the cell  $(m - i + 1, n - j + 1)$ . Since each lattice path  $P$  is either above the cell  $(i, j)$  or below the cell  $(i, j)$ , we have

$$f(i, j) + f(m - i + 1, n - j + 1) = |\mathcal{P}(B_{mn})|.$$

But the number of lattice paths in  $\mathcal{P}(B_{mn})$  is  $\binom{m+n}{m}$ , we get

$$f(i, j) + f(m - i + 1, n - j + 1) = \binom{m+n}{m}. \quad (2.16)$$

Summing (2.16) over  $(i, j)$  gives

$$2 \sum_{1 \leq i \leq m, 1 \leq j \leq n} f(i, j) = \binom{m+n}{m} mn.$$

This completes the proof. ■

**Lemma 2.5** *For positive integers  $m$  and  $n$ , we have*

$$\sum_{1 \leq i \leq m, 1 \leq j \leq n} if(i, j) = \binom{m+2}{3} \binom{m+n}{m+1} \quad (2.17)$$

and

$$\sum_{1 \leq i \leq m, 1 \leq j \leq n} jf(i, j) = \binom{n+2}{3} \binom{m+n}{n+1}. \quad (2.18)$$

*Proof.* Let

$$G(m, n) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} if(i, j).$$

To prove (2.17), we establish a recurrence relation for  $m, n \geq 2$ ,

$$G(m, n) = G(m-1, n) + G(m, n-1) + \binom{m+1}{2} \binom{m+n-1}{m}. \quad (2.19)$$

In doing so, let  $T$  be the set of triples  $(P, C_1, C_2)$ , where  $P$  is a path in  $\mathcal{P}(B_{mn})$ ,  $C_1$  and  $C_2$  are cells above  $P$  and are in a same column with  $C_2$  not lower than  $C_1$ . Notice that  $C_1$  and  $C_2$  are allowed to the same cell.

We proceed to compute  $|T|$  in two ways. First, it is easily seen that  $if(i, j)$  is the number of triples in  $T$  with  $C_1 = (i, j)$ . Hence we have for  $m, n \geq 1$ ,  $|T| = G(m, n)$ .

Alternatively,  $|T|$  can be computed as follows.

For a given lattice path  $P$  in  $\mathcal{P}(B_{mn})$ , the cells above  $P$  form a Ferrers diagram of a partition, denoted by  $\mu$ . Let  $\mu'$  be the conjugate of  $\mu$ , that is, there are  $\mu'_j$  cells in the  $j$ -th column of the Ferrers diagram of  $\mu$ .

In the  $j$ -th column of the Ferrers diagram of  $\mu$ , there are  $\binom{\mu'_j+1}{2}$  ways to choose  $C_1$  and  $C_2$  such that  $C_2$  is not lower than  $C_1$ . It follows that for given  $P$ , there are  $\sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}$  choices for  $C_1$  and  $C_2$ . Consequently, for  $m, n \geq 1$ ,

$$|T| = \sum_{\mu: 1 \leq \mu_1 \leq n, 1 \leq \mu'_1 \leq m} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}. \quad (2.20)$$

Hence, for  $m, n \geq 1$ ,

$$G(m, n) = \sum_{\mu: 1 \leq \mu_1 \leq n, 1 \leq \mu'_1 \leq m} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}. \quad (2.21)$$

For  $m, n \geq 2$ , the right hand side of (2.21) equals

$$\sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j + 1}{2} + \sum_{\mu: 1 \leq \mu_1 \leq n, 1 \leq \mu'_1 \leq m-1} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j + 1}{2}. \quad (2.22)$$

It is evident from (2.21) that the second double sum in (2.22) can be expressed by  $G(m-1, n)$ . The first double sum in (2.22) can be rewritten as

$$\sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \sum_{2 \leq j \leq \mu_1} \binom{\mu'_j + 1}{2} + \sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \binom{m+1}{2}. \quad (2.23)$$

Clearly, the number of partitions  $\mu$  with  $1 \leq \mu_1 \leq n$  and  $\mu'_1 = m$  equals the number of lattice paths from the lower-left corner to the upper-right corner in  $B_{m,n-1}$ , which is  $\binom{m+n-1}{m}$ . Hence the second sum in (2.23) simplifies to

$$\binom{m+1}{2} \binom{m+n-1}{m}. \quad (2.24)$$

To compute the double sum in (2.23), let  $\tilde{\mu}$  denote the partition obtained from  $\mu$  by deleting the first column of the Ferrers diagram of  $\mu$ . So we see that

$$\sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \sum_{2 \leq j \leq \mu_1} \binom{\mu'_j + 1}{2} = \sum_{\tilde{\mu}: 0 \leq \tilde{\mu}_1 \leq n-1, \tilde{\mu}'_1 \leq m} \sum_{1 \leq j \leq \tilde{\mu}_1} \binom{\tilde{\mu}'_j + 1}{2}. \quad (2.25)$$

From (2.21) it can be seen that the right hand side of (2.25) equals  $G(m, n-1)$ . Combining (2.21)–(2.25), we arrive at the recurrence relation (2.19).

For  $m, n \geq 1$ , let

$$F(m, n) = \binom{m+2}{3} \binom{m+n}{m+1}.$$

To prove that  $G(m, n) = F(m, n)$  for  $m, n \geq 1$ , it is sufficient to check that  $F(m, n)$  satisfies the same recurrence relation (2.19) and the same initial conditions. Clearly,  $F(1, n) = G(1, n)$  and  $F(m, 1) = G(m, 1)$  for  $m, n \geq 1$ . Moreover, it is easily checked that the recurrence relation (2.19) holds for  $F(m, n)$  as well. This proves identity (2.17). Relation (2.18) can be viewed as a restatement of (2.17). This completes the proof.  $\blacksquare$

Now we are ready to prove the conjecture of Armstrong, Hanusa and Jones on the average size of a self-conjugate  $(s, t)$ -core.

*Proof.* Let  $SC(s, t)$  denote the set of self-conjugate  $(s, t)$ -cores. We aim to show that

$$\sum_{\lambda \in SC(s, t)} |\lambda| = \frac{(s+t+1)(s-1)(t-1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}. \quad (2.26)$$

By Theorem 2.2, we find that

$$\sum_{\lambda \in SC(s, t)} |\lambda| = \sum_{P \in \mathcal{P}(A)} |\Phi(P)|. \quad (2.27)$$

Using Lemma 2.3, we obtain that

$$\sum_{P \in \mathcal{P}(A)} |\Phi(P)| = \frac{(s^2 - 1)(t^2 - 1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} - \sum_{P \in \mathcal{P}(A)} \sum_{(i,j) \text{ is above } P} A_{i,j}. \quad (2.28)$$

Combining (2.27) and (2.28), we see that

$$\sum_{\lambda \in SC(s,t)} |\lambda| = \frac{(s^2 - 1)(t^2 - 1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} - \sum_{P \in \mathcal{P}(A)} \sum_{(i,j) \text{ is above } P} A_{i,j}. \quad (2.29)$$

By the definition (2.1) of the array  $A$ , we deduce that

$$\begin{aligned} \sum_{P \in \mathcal{P}(A)} \sum_{(i,j) \text{ is above } P} A_{i,j} &= \sum_{P \in \mathcal{P}(A)} \sum_{(i,j) \text{ is above } P} (st + s + t - 2sj - 2ti) \\ &= (st + s + t) \sum_{1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor} f(i, j) - 2s \sum_{1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor} jf(i, j) \\ &\quad - 2t \sum_{1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor} if(i, j). \end{aligned} \quad (2.30)$$

Applying Lemma 2.4 and Lemma 2.5 to (2.30) with  $m = \lfloor \frac{s}{2} \rfloor$  and  $n = \lfloor \frac{t}{2} \rfloor$ , we get

$$\begin{aligned} \sum_{P \in \mathcal{P}(A)} \sum_{(i,j) \text{ is above } P} A_{i,j} &= (st + s + t) \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} \frac{\lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor}{2} - 2s \binom{\lfloor \frac{t}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor - 1} \\ &\quad - 2t \binom{\lfloor \frac{s}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor - 1}. \end{aligned} \quad (2.31)$$

We claim that

$$\begin{aligned} \frac{(s^2 - 1)(t^2 - 1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} &= \frac{(s + t + 1)(s - 1)(t - 1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} \\ &\quad + (st + s + t) \frac{\lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor}{2} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} \\ &\quad - 2t \binom{\lfloor \frac{s}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor - 1} - 2s \binom{\lfloor \frac{t}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor - 1}, \end{aligned}$$

which simplifies to

$$\frac{st(s - 1)(t - 1)}{24} = (st + s + t) \frac{\lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor}{2} - \frac{t}{3} (\lfloor \frac{s}{2} \rfloor + 2) \lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor - \frac{s}{3} (\lfloor \frac{t}{2} \rfloor + 2) \lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor.$$

When  $s$  and  $t$  are coprime, at least one of  $s$  and  $t$  is odd. Without loss of generality, we may assume that  $s$  is odd. In this case, it is easily checked that above relation is true. Thus the claim holds. Combining (2.29) and (2.31), we arrive at (2.26), and hence the proof is complete.  $\blacksquare$

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